Search Less for a Better Price

Eren Bilen[*], Deniz Dizdar[†] and Chun-Hui Miao[‡]

Abstract

In many markets, sellers must spend resources to learn the costs of providing goods/services. This paper considers consumer searches in such markets. It is found that (1) even with ex ante identical consumers and sellers, there is price dispersion in the equilibrium; (2) despite price dispersion and minimal search costs, it could be optimal to search just two sellers; (3) the optimal number of searches can increase with sellers' information costs. (JEL D40, L00)

Keywords: Price dispersion, Precontract costs, Search costs, Sealed-bid Auction.

[*]Department of Data Analytics, Dickinson College, Carlisle, PA 17013. Email: bilene@dickinson.edu

[†]Department of Economics, University of Montréal, C.P. 6128 succursale Centre-Ville, Québec H3C 3J7, Canada. Email: deniz.dizdar@umontreal.ca

[‡]Department of Economics, University of South Carolina, Columbia, SC 29208. Email: miao@moore.sc.edu

1 Introduction

To make informed purchase decisions, consumers search. To earn their business, sellers provide relevant information such as prices. The standard economic models of consumer search assume that price search is costly, but price-setting is costless.[1] In many markets, however, even a simple price quote may involve nontrivial costs for the seller. For example, a mortgage lender must evaluate a borrower's creditworthiness before offering a rate quote;[2] an insurance agent must assess an applicant's risk characteristics before issuing the premium; a repair shop must diagnose the problem before giving a cost estimate. In these markets, production costs depend on consumers' individual needs. Sellers set prices after consumer search takes place. A consumer can canvass multiple sellers, but cannot contract on any seller's effort in preparing the price quote.

This paper incorporates the above features into a model of consumer search where sellers observe how many other sellers the consumer has contacted with a request for a price quote. Our goal is to study the market impact of precontract costs, including consumer search cost and price-setting cost. The latter cost is due to uncertainty in the production cost, which we assume to be identical across sellers: each seller can learn the true production cost (before all sellers simultaneously make competing price offers) if he pays an "information cost" t.[3] Under price competition, the price quotes will reflect information costs. While this observation suggests an equivalence between consumer search and sellers' information acquisition — both costs are ultimately paid by the consumer — our analysis reveals an important difference: the only way to save on total search costs is to search less, but whether the consumer can obtain a better expected price by searching just two sellers or by searching more sellers depends on the level of the information cost. Consequently, information costs and search costs can have different, even opposite, impacts on consumer search behavior. Searching a large number of sellers yields a lower expected price if sellers' willingness to incur information costs drops sharply when they face more competitors, which turns out to be the case for an intermediate range of, relatively high, levels of the information cost.

[1] It is a long tradition that began with Stigler's seminal paper (Stigler 1961). More recently, a class of search models with an "information clearinghouse" assume nearly the opposite, that is, zero (marginal) search cost but positive (fixed) advertising cost (Baye and Morgan 2001).

[2] Woodward (2008) estimates that the "dry-hole" - applications that are processed but fail to become loans - cost of a mortgage loan is somewhere in the range of \$120 to \$410.

[3] It should be distinguished from the so-called menu costs, originally introduced by Sheshinski and Weiss (1977). While menu costs are unavoidable for every price change, sellers in our model can avoid the information costs should they choose not to acquire information.

Interestingly, however, despite price dispersion and even for very low or zero search costs, searching just two sellers is optimal for a broad range of, either sufficiently small or sufficiently large, information costs. The model thus provides a possible explanation for why consumers appear to undertake surprisingly little search in relevant markets.[4]

This paper contributes to the understanding of transaction costs. Dahlman (1979) classifies transaction costs into three categories based on the stages of a contract: search and information costs (precontract), bargaining and decision costs (contract), policing and enforcement costs (post-contract). While the impact of consumer search costs on market outcomes has been extensively studied,[5] their interaction with sellers' information costs has so far received scant attention. A notable exception is French and McCormick (1984), whose informal analysis of the service market anticipates many of the themes explored in this paper. After showing that the winning bidder's expected profit equals the sum of his competitors' sunk costs of bid preparation under a free-entry condition, they argue that consumers indirectly pay for service providers' information costs. The focus of their paper, however, is on firms' marketing strategies, such as how likely firms charge for their estimates or advertise, whereas the focus of this paper is on the problem faced by the consumer side.[6]

Pesendorfer and Wolinsky (2003) and Wolinsky (2005) have also considered consumer search in the presence of information costs. Assuming that service outcomes are not contractible (but price searches are costless), Pesendorfer and Wolinsky (2003) examine market inefficiencies when a consumer must rely on second opinions to pick the right contractor. Under the assumption that sellers can provide better matching via costly investments, Wolinsky (2005) shows that consumers' inability to internalize sellers' costs leads to excessive search. Despite the similarity, these two papers have a different focus than ours: they are concerned with prior information on product character-

[4]Lee and Hogarth (2000) find that a majority of mortgage borrowers consult less than three lenders or brokers. Woodward and Hall (2012) argue that the pecuniary search costs for mortgage loans implied by the number of searches are implausibly high. See also Honka (2014) for evidence from the US auto insurance market, Allen, Clark, and Houde (2014) and Alexandrov and Koulayev (2017) from the US and Canadian mortgage markets, respectively, and Stango and Zinman (2015) from the credit card market.

[5]See Anderson and Renault (2017) for a recent survey of consumer search theories.

[6]Due to the lack of formal game-theoretic analysis, the connection between assumptions and results is somewhat opaque in their paper. For example, it is not clear whether the predicted pattern is the result of collective behavior among sellers or the noncooperative outcome.

istics (so there is no price dispersion), whereas our paper is concerned with prior information on prices. Because of this difference, our results on prices do not exist in their models.[7]

The current paper assumes that the consumer can commit to any chosen number of searches. A commitment is possible if the number of price requests is observed by sellers.[8] This assumption is satisfied reasonably well in the aforementioned markets. For example, in the mortgage market, lenders can infer a consumer's search intensity from the number of credit inquiries recorded in the consumer's credit report.[9] The same is true in the market for auto, home, and life insurances.[10] Admittedly, in other markets, the number of consumer searches may not be observable. We view our current analysis of the commitment case as providing a useful benchmark. In a companion paper, Miao (2020) studies consumer search behavior in a similar setting, but the number of consumer searches is not publicly observable. It is found that, in the absence of the ability to commit, consumers may be worse off when search costs decline. The two papers are complementary and apply to different markets.[11]

Our paper is also related to Burdett and Judd (1983) ("BJ-83"). In both papers, consumers engage in fixed-sample size searches of ex-ante identical sellers. However, they assume costless price-setting. Because of this, consumer welfare is maximized when everyone searches exactly twice, with prices being set at marginal costs. Moreover, price dispersion disappears when search costs approach zero. These results are different from those of this paper.[12]

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 presents some preliminary results, including a complete characterization of sellers' equilibrium behavior, for every number of searches. Section 4 contains our main results on the optimal number of searches and the equilibrium impact of precontract costs. Section 5 concludes. All proofs are given in the appendix.

[7] For tractability, these two models make two assumptions that are somewhat unrealistic: (1) prices are set before diagnoses; (2) search costs are not paid until a contract offer is accepted.

[8] See Bagwell (1995) for a classic discussion on the relationship between commitment and observability.

[9] A bank's cost of pulling consumer credit is small relative to other costs during the loan application process, so is assumed to be negligible in this paper.

[10] During insurance application interviews, insurance companies often ask whether an applicant has other applications pending (`https://www.nerdwallet.com/article/insurance/life-insurance-application`). The answer to this question provides another indicator of the applicant's search intensity.

[11] Another difference is that this paper explicitly models sellers' decisions on whether to acquire information, whereas Miao (2020) follows Lang and Rosenthal (1991) by adopting a reduced-form assumption of bidding costs. Because of this difference, the current paper obtains a richer set of results on the optimal number of searches, which could not have been obtained for a reduced-form specification.

[12] See Section 4 for more details.

2 The Model

A consumer is willing to pay v for a good or service (henceforth, the product), which can be provided by any one of N sellers. All sellers face the same exogenously given production cost c. This symmetry assumption ensures that equilibrium price dispersion cannot be attributed to different cost realizations across sellers. It is also consistent with empirical research in related markets.[13] The production cost c is equal to c_l with probability $q \in (0,1)$ and equal to c_h with probability $1-q$, where $0 \leq c_l < c_h < v$ (the last inequality implies that the social value of a trade is always positive, and ensures that the consumer buys something in equilibrium).

To find the best deal, the consumer may request price quotes from different sellers. The cost of each request (the "search cost") is $s \geq 0$. A seller who has received a request can learn the actual value of c (before quoting a price) if he pays the "information cost" $t \geq 0$. The buyer cannot learn c. This assumption is made on the grounds that sellers have more expertise than consumers in the relevant markets. It also ensures that sellers do not face an adverse selection problem.[14] The values of v, c_l, c_h, q, s and t are common knowledge. Both the consumer and the sellers are risk-neutral.

We study the (mixed-strategy) sequential equilibria ("SE") of the following game between the consumer, who seeks to minimize her expected total expense (the sum of the total search costs and the price paid for the product), and the sellers, who seek to maximize their expected profits.

1. Chance decides whether c is equal to c_l or equal to c_h.

2. The consumer decides which sellers to contact with a request for a price quote. We let n denote the number of contacted sellers.

3. All contacted sellers observe how many sellers have received a request, and then decide whether to privately learn c and which price to quote (depending on either the acquired or the prior information about c).

[13] For example, in her study of the auto insurance market, Honka (2014) reports that over 93% of consumers kept their coverage choice the same during the last shopping occasion and were searching only for the lowest premium. Similarly, in their study of the Canadian mortgage market, Allen et al. (2014) find that contracts are homogeneous, and for a given consumer costs are mostly common across lenders due to loan securitization and a government insurance program.

[14] In a model with a similar setup, Lauermann and Wolinsky (2017) assume that an auctioneer has private information, which affects the number of bidders she solicits. Their paper, however, does not consider *bidders*' precontract costs.

4. After receiving all price quotes, the consumer decides which offer to accept, if any.

Clearly, if there is a unique lowest price offer (below v), the consumer will choose that offer. As is standard in the literature, we assume that if multiple sellers offer the same lowest price the consumer selects one of these offers uniformly at random, and restrict attention to symmetric equilibria in which all sellers choose the same strategy: for each value of n, the behavior of each contacted seller is described by the same quadruple $(\alpha(n), F_l(\cdot|n), F_h(\cdot|n), F_b(\cdot|n))$, where $\alpha(n)$ is the probability that the seller acquires information about the production cost, $F_b(\cdot|n)$ is the c.d.f. of price quotes chosen by the seller if he remains uninformed, and $F_i(\cdot|n)$, $i \in \{l, h\}$, is the c.d.f. of price quotes for the case that the seller has learned that the production cost is c_i.

Note that, by assumption, the consumer is engaged in a fixed-sample size ("FSS") search, as opposed to a sequential search where the consumer visits sellers one-by-one and stops search once her reservation price is met. We adopt this approach for two reasons: first, existing empirical evidence suggests that FSS search provides a more accurate description of observed consumer search behavior (De Los Santos, Hortaçsu, and Wildenbeest 2012, Honka and Chintagunta 2017); second but particularly relevant to this model, costly information acquisition can cause delay and a delay is a more significant problem for sequential search than for FSS search.[15] For example, in the US mortgage market, a consumer typically receives a Loan Estimate three business days after the initial request,[16] but a lender is only required to honor the terms of a Loan Estimate for ten business days.[17] Therefore, it may actually be optimal for a consumer to request price quotes from several lenders at once, rather than one after another. For the same reason, we assume that the consumer cannot engage in multiple rounds of search.[18]

We assume that s is much smaller than v, so that it is never optimal for the consumer to search just one seller (in which case she minimizes the total search costs ns but has to pay a monopoly price for the product). Moreover, we assume that the pool of sellers is so large that it never constrains the consumer's number of searches unless the consumer would like to contact infinitely

[15] In the same vein, Morgan and Manning (1985) and Janssen and Moraga-González (2004) argue that fixed-sample size search is more appealing when a consumer needs to gather price information quickly.

[16] "*Loan Estimate and Closing Disclosure: Your guides in choosing the right home loan*", Nicole Shea, Consumer Financial Protection Bureau (CFPB), Aug 19, 2019.

[17] "*Real Estate Settlement Procedures Act*", CFPB Consumer Laws and Regulations, March 2015.

[18] While it appears that the consumer could choose to search again after extracting information from early rounds of offers, doing so would lower sellers' incentive to acquire information and render their bids uninformative, defeating the very purpose of searching multiple rounds.

many sellers (in the latter case, which may happen only if $s = 0$, the finite number of sellers is needed for equilibrium existence). These assumptions allow us to focus only on nontrivial cases and to avoid purely technical case distinctions.

3 Preliminary Results

3.1 A Benchmark Result

A useful point of departure is to consider what happens if $t = 0$, i.e., if sellers can costlessly learn the production cost. This is not only the assumption of a frictionless market, but also the working assumption of almost all consumer search models. In this case, since sellers have the same production cost, based on the standard Bertrand style argument, we can see that the prices will be set equal to the (realized) production cost as long as there are at least two sellers. The consumer cannot do better by visiting more than two sellers because she cannot get a better offer, nor can she do better by visiting just one seller, who will charge a monopoly price. In equilibrium, the consumer earns an expected surplus of $v - c_E - 2s$, where $c_E = qc_l + (1-q)\,c_h$ is the expected production cost. This serves as a natural benchmark for the current analysis.

The same outcome could be obtained even for positive information cost if the consumer and sellers were able to contract on information acquisition efforts : acquiring information about the production cost allows sellers to earn information rents, but it is purely wasteful from a social point of view.[19] Therefore, the consumer would prevent sellers from earning information rents by simply requiring sellers not to acquire product cost information. The sellers would again compete à la Bertrand, with each of them quoting a price of c_E and earning zero profits in equilibrium, and the consumer would earn the same expected surplus as in the benchmark. Albeit straightforward, this result demonstrates that the information cost, in itself, does not necessarily cause a welfare loss for the consumer. Rather, any such loss is due to the inability to contract on sellers' information acquisition efforts. Of course, if the information cost is so high that it exceeds sellers' private value of information, which equals $q\,(1-q)\,(c_h - c_l)$, then no seller will learn c (even if he is not

[19]It should be noted that this is true because we assume $c_h < v$. Otherwise, information acquisition will not be a pure waste. We are grateful to an anonymous reviewer for making this point. To analyze the case where $c_h > v$ will require us to significantly expand the game and is thus beyond the scope of this paper.

contractually precluded from doing so), it is optimal for the consumer to obtain just two competing price quotes, and the expected consumer surplus is again $v - c_E - 2s$.

In all the above cases, consumer surplus is maximized when the consumer obtains two competing price quotes.[20] We summarize these observations as follows:

Observation 1 *The number of searches in the unique symmetric SE is two and the expected consumer surplus is* $v - c_E - 2s$ *if* $t = 0$ *or* $t \geq q(1-q)(c_h - c_l)$*. The same would obtain for all values of* t *if the consumer and sellers could contract on information acquisition efforts.*

For the remainder of the paper, we focus on the more interesting case for which t is positive but not so large that it is never optimal for sellers to acquire information, i.e., $t \in (0, q(1-q)(c_h - c_l))$.

3.2 Sellers' equilibrium strategy

As mentioned in Section 2, we look for symmetric equilibria where the consumer selects an offer uniformly at random if she receives multiple lowest offers, and where all sellers use the same strategy $n \mapsto (\alpha(n), F_l(\cdot|n), F_h(\cdot|n), F_b(\cdot|n))$.

Lemma 1 *For any* $t \in (0, q(1-q)(c_h - c_l))$*, any symmetric SE and any* $n \geq 2$*,* $\alpha(n) \in (0,1)$*.*

Lemma 1 implies that sellers always randomize between submitting an informed quote and submitting a blind quote. Our first key result, Lemma 2 below, characterizes the unique symmetric SE strategy for sellers, which involves not only randomized learning about the production cost but also randomized price quotes.

Lemma 2 *For any* $t \in (0, q(1-q)(c_h - c_l))$*, there is a symmetric SE, and any symmetric SE features the same equilibrium strategy* $n \mapsto (\alpha^*(n), F_l^*(\cdot|n), F_h^*(\cdot|n), F_b^*(\cdot|n))$ *for sellers. For any*

[20] It goes without saying that the result will be different if the sellers do not have the same production cost. In MacMinn (1980) and Spulber (1995), sellers' price setting is equivalent to bidding in a private value auction. Price dispersion arises from cost heterogeneity of sellers. Alternatively, even if sellers have the same cost, consumer heterogeneity in captivity leads to an asymmetric mixed strategy Nash equilibrium (Gilgenbach 2015).

$n \geq 2$, we have

$$\alpha^*(n) = 1 - \left(\frac{t(1-q)}{q((c_h - c_l)(1-q) - t)} \right)^{1/(n-1)},$$

$$F_h^*(p|n) = 0 \quad \textit{for} \quad p < c_h \quad \textit{and} \quad F_h^*(p|n) = 1 \quad \textit{for} \quad p \geq c_h,$$

$$F_l^*(p|n) = \frac{1 - \left(\frac{t}{q(p-c_l)} \right)^{1/(n-1)}}{\alpha^*(n)} \quad \textit{for} \quad p \in [\underline{p}_l^*, \bar{p}_l^*] = [c_l + t/q, c_h - t/(1-q)],$$

$$F_b^*(p|n) = 1 - \frac{\alpha^*(n)}{1-\alpha^*(n)} \left(\left(\frac{q(p-c_l)}{(1-q)(c_h-p)} \right)^{\frac{1}{n-1}} - 1 \right)^{-1} \quad \textit{for} \quad p \in [\underline{p}_b^*, \bar{p}_b^*] = [\bar{p}_l^*, c_h],$$

and the expected price paid by the consumer is $c_E + n\alpha^*(n)t$.

Thus, sellers who learn that the production cost is c_h quote price c_h, uninformed sellers randomize over prices between $c_h - t/(1-q)$ and c_h according to the atomless distribution $F_b^*(\cdot|n)$, and sellers who learn that the production cost is c_l quote even lower prices, randomizing according to the atomless c.d.f. $F_l^*(\cdot|n)$ over prices between $c_l + t/q$ and $c_h - t/(1-q)$ (the lowest price offered by uninformed sellers). Moreover, the consumer pays for the total information costs $n\alpha^*(n)t$, in the form of a higher expected price.

Figure 1 illustrates (for $q = 1/2$) how the information cost affects price distributions. The red solid curve depicts the price distribution when $t = 0.2(c_h - c_l)$ and the black dashed curve when $t = 0.1(c_h - c_l)$. For each level of information cost, there are two segments of price distributions, corresponding to informed bids (in state l) and blind bids.

From the graph, we can see that sellers are more likely to set low blind bids when t is large. Intuitively, a larger t makes sellers less likely to acquire costly information and this means uninformed sellers are less likely to suffer from the winner's curse. As a result, uninformed sellers bid more aggressively. The effects of a larger t on the bidding behavior of a seller informed of a low cost, however, are more complicated: on the one hand, having fewer competing bids by informed sellers raises the lower bound of informed bids (in a way that allows informed sellers to exactly recoup the information cost); on the other hand, more aggressive bidding by uninformed sellers

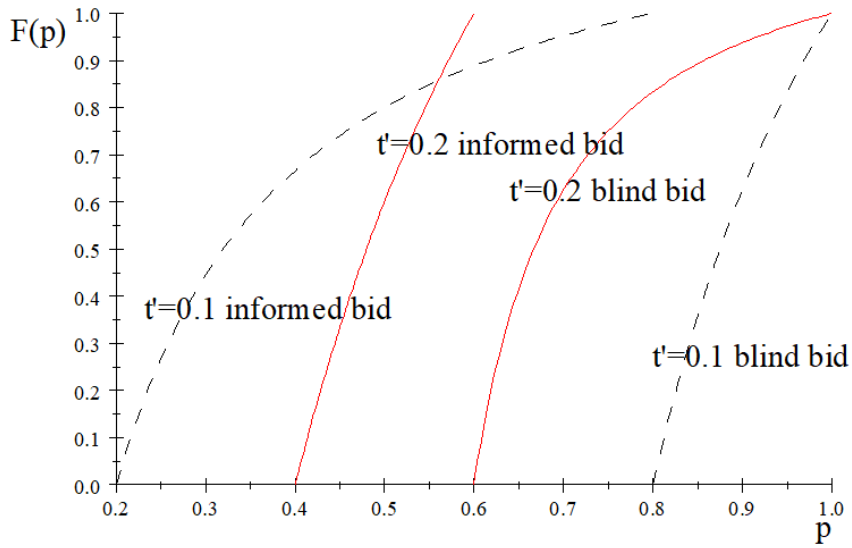

Figure 1: $(q = 1/2)$ The red solid curve depicts the price distribution when $t' = t/(c_h - c_l) = 0.2$ and the black dashed curve when $t' = 0.1$.

lowers the upper bound of informed bids. Therefore, a larger t decreases the degree of dispersion in informed bids.[21] These observations are summarized in Lemma 3:

Lemma 3 (i) *The range of informed bids decreases with* t; (ii) *The range of blind bids increases with* t; *moreover, blind bids for* t_1 *first-order stochastically dominate blind bids for* t_2 *if* $t_1 < t_2$.

4 The Optimal Number of Searches

Lemma 2 shows that when sellers acquire information, i.e., for $t \in (0, q(1-q)(c_h - c_l))$, the consumer pays the total information costs $n\alpha^*(n,t)t$ indirectly in the form of a higher expected price.[22] This explains why the expected price depends on the number of searches, and gives us a basic intuition for why prices might sometimes increase with the number of searches, so that the consumer may want to limit this number to get a better price (on top of the incentive to save on search costs): even though $\alpha^*(n,t)$ decreases with n (if there is more competition, each individual seller has a lower

[21] A standard measure of dispersion is the variance in prices. Unfortunately, we do not have an analytical solution for the variance. Other commonly used metrics to measure price dispersion include the range of prices (Baye, Morgan, and Scholten 2006), which is the one used here.

[22] We highlight the dependence of α^* on t explicitly in the notation from now on.

incentive to acquire information to earn information rents), the total information costs $n\alpha^*(n,t)t$, and hence the expected price, might increase with n.

The equilibrium consumer surplus can be written as $v - c_E - \min_{n\geq 2}(\alpha^*(n,t)t + s)n$. Relative to the benchmark case, it is lower by $\min_{n\geq 2}(\alpha^*(n,t)t + s)n - 2s$. The term $\psi(n,s,t) := (\alpha^*(n,t)t + s)n$ captures the overall impact of precontract costs, including consumer search cost and seller information cost, on consumer welfare. It does not contribute to sellers' profit margin and is merely a waste caused by market frictions, but for want of a better name we shall call it the *expected markup*.

To determine the equilibrium, or optimal, number n^o of searches by the consumer, we start by examining how the expected markup (essentially the negative of consumer surplus) varies qualitatively with the number of searches, depending on the parameters s and t. Due to its technical nature, we relegate the details of this analysis to the appendix (see Observation 2, Observation 3, and Lemma 4) and merely summarize the results in what follows. For ease of exposition, and to be able to use the tools of calculus, we follow Wolthoff (2017) and treat n as a continuous variable. That is, we define the expected markup, using the formula for $\alpha^*(n,t)$ from Lemma 2, for all real numbers $n \in (1,\infty)$. Our subsequent characterization of the optimal number of searches n^o, in Proposition 1 below, will still ignore the integer constraint, but will take the constraint $n \geq 2$ into account. We will also explain the (straightforward) consequences of Proposition 1 for the case where n is again required to be an integer.

According to Lemma 4, there are three possible patterns of how the expected markup varies with the number of searches. Figure 2 illustrates these possibilities. When the information cost t is small (the blue dashed curve at the bottom), the expected markup monotonically increases with n. When t is large but s is zero (the black dotted curve in the middle), the total information cost, and hence the expected markup, is *unimodal* (i.e., first increasing and then decreasing) in n. When t is large and s is positive (the red solid curve at the top), the expected markup increases up to a local maximum $n_1 = n_1(s,t)$, then decreases up to a local minimum $n_2 = n_2(s,t)$, and then increases again for $n > n_2$.

Hence, if $s > 0$, there are at most two candidates for the optimal number of price quotes n^o, i.e., the solution to the problem $\min_{n\in[2,N]} \psi(n,s,t)$, namely $n = 2$ and $n = n_2$ (recall that we maintain the constraint $n \geq 2$ but ignore the integer constraint here, and that N is assumed sufficiently

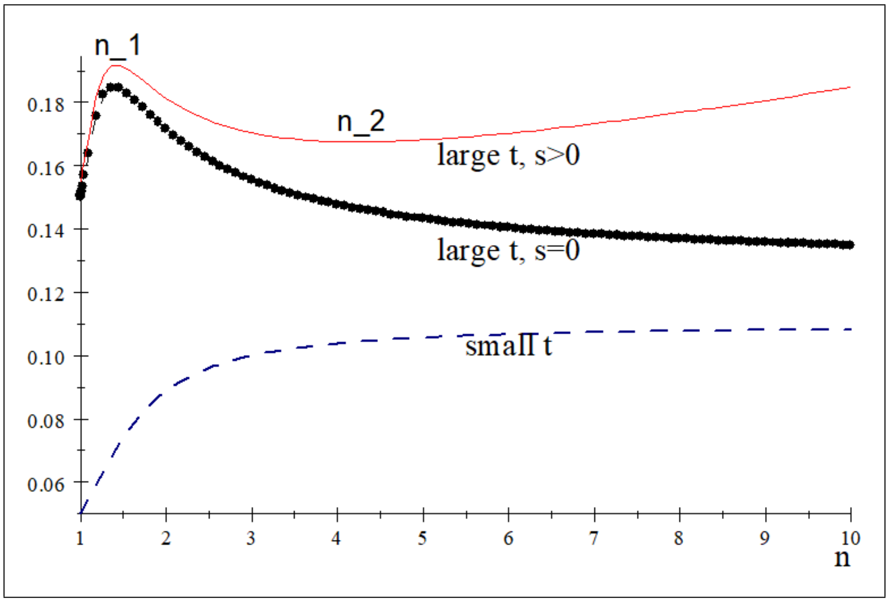

Figure 2: ($q = 1/2$) The expected markup as a function of n. Blue dashed ($t = 0.05(c_h - c_l), s = 0$), Black dotted ($t = 0.15(c_h - c_l), s = 0$), Red solid ($t = 0.15(c_h - c_l), s = 0.005(c_h - c_l)$).

large so as not to constrain the consumer's choice), and n^o can in principle be determined by comparing $\psi(2, s, t)$ and $\psi(n_2, s, t)$, and by checking whether n_2 actually satisfies the constraint $n_2 \geq 2$. This task is complicated by the fact that there is no analytical solution for $n_2(s, t)$. Still, our detailed study of the expected markup function in Lemma 4, combined with a few additional arguments, allows us to make the relevant comparison (and thus to characterize when $n^o = 2$ and when $n^o > 2$) for all values of t and all $s > 0$. Similarly, if $s = 0$, the only candidates for n^o are $n = 2$ and $n = N$. Assuming once again that N is very large, n^o can be determined by comparing $\psi(2, s, t)$ and $\lim_{n\to\infty} \psi(n, s, t)$, which (given Lemma 4) is a fairly straightforward task.

To state Proposition 1, we define

$$\gamma(t) := -\ln \frac{t(1-q)}{q((c_h - c_l)(1-q) - t)}$$

for all $t \in (0, q(1-q)(c_h - c_l))$. Note that the function γ is strictly decreasing and satisfies $\lim_{t\to 0} \gamma(t) = +\infty$ and $\lim_{t\to q(1-q)(c_h-c_l)} \gamma(t) = 0$. Its inverse, γ^{-1}, satisfies $\gamma^{-1}(x) = \frac{q(1-q)(c_h-c_l)}{(1-q)e^x+q}$ for all $x > 0$. The cutoffs $t^*(s), t_1(s), t_2(s)$, and $\bar{s}$ occurring in the statement of Proposition 1 are characterized explicitly in Observation 3 in the appendix and satisfy the following properties. For all $s \in (0, \bar{s})$, we have $0 < t^*(s) \leq t_1(s) < t_2(s) < q(1-q)(c_h - c_l)$, with $t^*(s) = t_1(s)$ if and only if

$\gamma(t^*(s)) = 1$. Moreover, t^* is continuous and strictly increasing (i.e., $\gamma(t^*(s))$ is strictly decreasing in s), with $\gamma(t^*(0)) = 2$ and $\gamma(t^*(\bar{s})) < 1$, t_1 is strictly increasing and t_2 is strictly decreasing.

Proposition 1 *(i) If $s = 0$, then there is a unique value $\hat{t}(0)$, satisfying $\gamma(\hat{t}(0)) \approx 1.594$, (i.e., $\frac{\hat{t}(0)}{c_h - c_l} \approx \frac{q(1-q)}{4.92(1-q)+q}$), such that $n^o = 2$ if $t < \hat{t}(0)$, whereas $n^o = N$ if $t > \hat{t}(0)$.*

(ii) If s is positive but small enough for $\gamma(t^*(s)) > 1$ to hold, then $n^o = 2$ if $t < t^*(s)$ or $t > t_2(s)$, and $n^o > 2$ if $t \in (t_1(s), t_2(s))$. Furthermore, we have $n^o = 2$ for some values $t \in (t^*(s), t_1(s))$ (including all values sufficiently close to $t^*(s)$), and $n^o > 2$ for other values in $t \in (t^*(s), t_1(s))$ (including all values sufficiently close to $t_1(s)$).[23]

(iii) If s is large enough for $\gamma(t^*(s)) \leq 1$ to hold but smaller than $\bar{s}$, then $n^o > 2$ if $t \in (t_1(s), t_2(s))$, and $n^o = 2$ if $t < t_1(s)$ or $t > t_2(s)$.

(iv) If $s \geq \bar{s}$, $n^o = 2$ for all $t \in (0, q(1-q)(c_h - c_l))$.

Proposition 1 shows that the optimal number of searches is "often" two, [24] but it also reveals when searching more than two sellers is optimal: this is the case if the search cost is not too large (below $\bar{s}$) and the information cost is in a "mid-range" (equal to $(\hat{t}(s), t_2(s))$ for small values of s, and equal to $(t_1(s), t_2(s))$ for larger values of s). In particular, the optimal number of searches is non-monotonic in t. The result is visualized in Figure 3 (for $q = 1/2$), which shows in particular how the range of information costs for which $n^o > 2$ shrinks as s increases. The region will of course be even smaller if the integer constraint for n is taken into account.

At first glance, the result that a consumer only needs to search twice in a market of homogeneous sellers may not be surprising. For example, the same result holds in BJ-83 based on the standard Bertrand style argument. However, there is a crucial difference. In BJ-83, if a consumer searches twice, then there will be no price dispersion, eliminating the need for further search. In the present model, a consumer searches twice despite price dispersion, because additional searches would change sellers' bidding strategies, potentially raising prices. It follows that, if search costs are zero, the "law of one price" will hold in BJ-83, but not in the present model.

[23]In the proof of Proposition 1, we give the precise condition that determines whether $n^o = 2$ or $n^o > 2$ for a given $t \in (t^*(s), t_1(s))$ when $\gamma(t^*(s)) > 1$. Numerically, we then find (non-surprisingly) that there is a unique value $\hat{t}(s) \in (t^*(s), t_1(s))$ such that $n^o = 2$ if $t < \hat{t}(s)$, and $n^o > 2$ if $t > \hat{t}(s)$. Thus, if $s > 0$ and $\gamma(t^*(s)) > 1$, we find that $n^o > 2$ if $t \in (\hat{t}(s), t_2(s))$, and $n^o = 2$ for $t < \hat{t}(s)$ or $t > t_2(s)$.

[24]Honka (2014) finds that consumers get on average 2.96 quotes with the majority of consumers collecting two or three quotes when purchasing auto insurance policies.

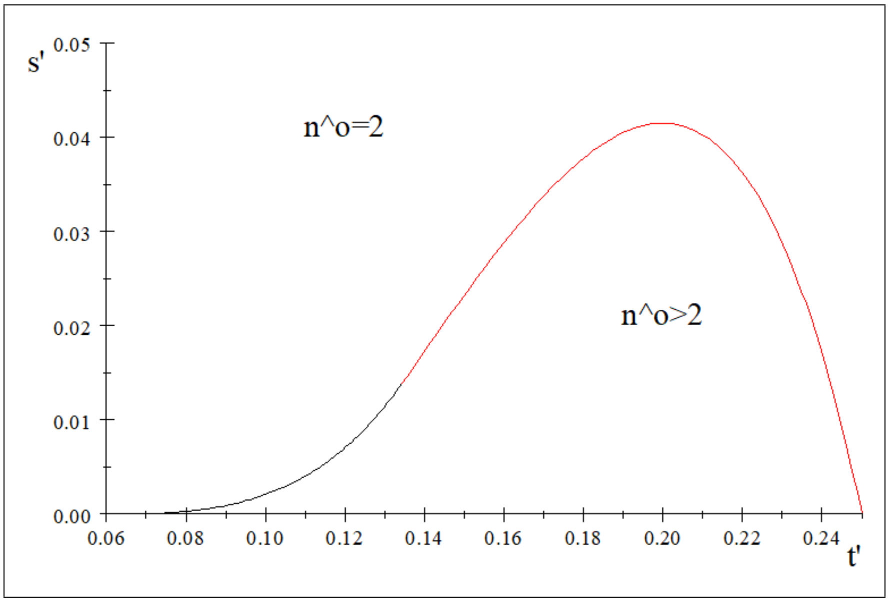

Figure 3: The Optimal Number of Searches, where $q = 1/2$, $t' = t/(c_h - c_l)$ and $s' = s/(c_h - c_l)$. The decreasing part of the red curve shows $t_2(s)$, the black part of the curve shows $\hat{t}(s)$, and the increasing part of the red curve shows $t_1(s)$ when $\gamma(t^*(s)) < 1$.

Our finding that searching two sellers is optimal for a broad range of information costs even if s is small is more similar to that of Che and Gale (2003), who find it optimal to include only two contestants in a research contest.[25] In their model, restricting entry to two competitors decreases the coordination problem of competing contestants and minimizes the duplication of fixed costs. Similarly, in the present model, limiting the number of bidders reduces duplication in wasteful information acquisition. However, unlike many other papers with a similar result, the present paper also shows that the consumer can sometimes benefit from expanding her search effort, especially when the search cost is small and the information cost is relatively large. The impact of the search cost is quite obvious, but the effect of the information cost is not. When the information cost rises, one might expect the consumer to search less since she has to indirectly pay for sellers' information costs, but this intuition is of course incomplete because it ignores the effect of an increase in n on sellers' incentive to acquire information. To provide further intuition for our findings, we plot $\alpha^*(n, t)/\alpha^*(2, t)$ in Figure 4. The graph illustrates two properties. First, sellers are less likely to acquire information when t is large. This is obvious. Less obvious is the other property, namely,

[25]A similar result is also found in auctions with entry (e.g., Harstad 1990, Levin and Smith 1994), and for research tournaments (Taylor 1995, Fullerton and McAfee 1999, Dizdar 2021).

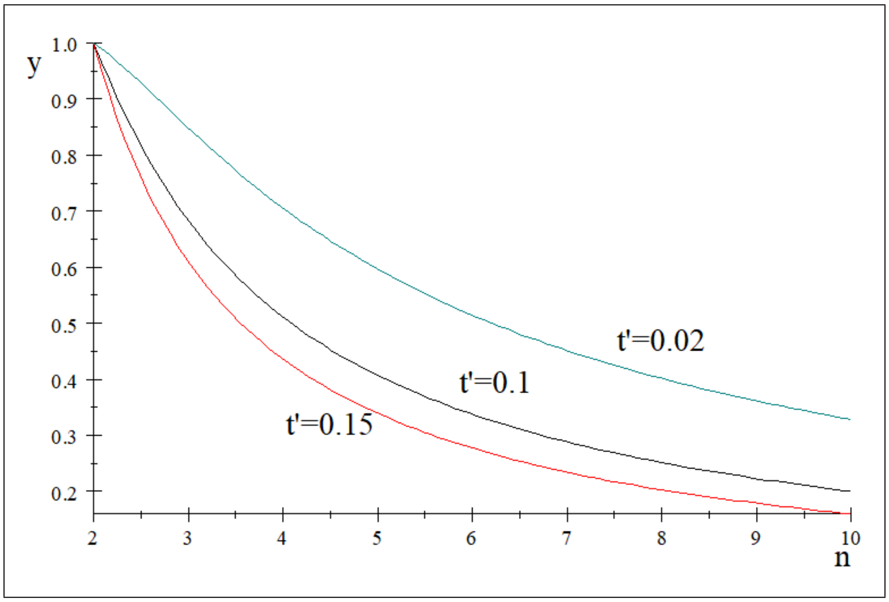

Figure 4: $y = \alpha^*(n,t)/\alpha^*(2,t)$ for $q = 1/2$ and $t' = t/(c_h - c_l) \in \{0.02, 0.1, 0.15\}$.

α^* decreases in n at a faster rate when t is large. This is why increasing the number of searches can reduce the wasteful and duplicative information acquisition efforts for larger values of t.

4.1 Comparative Statics

Here, we further examine the impact of search cost and information cost on the equilibrium outcome. We have already seen above that the range of values of t for which it is optimal to demand price quotes from more than two sellers shrinks as s increases. The next proposition shows furthermore that the optimal number of searches decreases with s.

Proposition 2 *For any $s < \tilde{s}$, if $n \in argmin_{n' \in [2,N]} \psi(n', s, t)$ and $\tilde{n} \in argmin_{n' \in [2,N]} \psi(n', \tilde{s}, t)$, then $\tilde{n} \leq n$.*

Combined with our previous finding that n^o is non-monotonic in t, Proposition 2 shows in particular that search costs and information costs have different impacts on consumers' search behavior, even though they both contribute to the overall precontract costs. This means that it is not only the total costs, but also the composition of the costs, that matter to consumer search. A similar result exists for two-sided markets, but the underlying mechanism is much different. In two-sided markets the composition of costs matters because the costs imposed on one side cannot be fully internalized by the other side of the market, whereas in this model it matters despite consumers' full internalization of sellers' costs.

Let $\varphi(s,t) := \psi(n^o(s,t), s, t)$ denote the equilibrium expected markup. Proposition 3 examines the welfare impact of the two costs.

Proposition 3 *The equilibrium expected markup $\varphi(s,t)$ increases with s, and is unimodal in t.*

Perhaps not surprisingly, the consumer benefits from a lower search cost. It is already clear, from our previous results that total information costs are zero and the consumer visits only two sellers if $t = 0$ or $t \geq q(1-q)(c_h - c_l)$, whereas there are positive amounts of (wasteful) information acquisition efforts for $t \in (0, q(1-q)(c_h - c_l))$, that the equilibrium expected markup is not monotonic in t. The additional insight here is the unimodality of $\varphi(s, \cdot)$, which we establish using a combination of analytical and numerical arguments.

4.2 Economic Significance of Information Costs

In order to evaluate the economic significance of information costs, we focus on the case $s = 0$, and compute the ratio between the equilibrium expected markup $\varphi(0,t) = n^o(0,t)\alpha^*(n^o(0,t),t)t$ and the consumer's expected total expense in the benchmark case without information costs, $c_E + 2s = c_E$.[26] Recall that $n^o(0,t)$ is either 2 or N.

Figure 5 plots the ratio between the expected markup if the consumer requests $n = 2$ price quotes and c_E ($\psi(2,0,t)/c_E$, the solid curve) and the corresponding ratio if the consumer requests "infinitely many" price quotes ($\lim_{n\to\infty} \psi(n,0,t)/c_E$, the dashed curve), for $q = 1/2$. The lower envelope of the two curves corresponds to $\varphi(0,t)/c_E$. As we can see from the graph, the existence of a small incontractible information cost (between 0 and $(c_h - c_l)/4$) can potentially increase the consumer's total expense by almost thirty percent. This demonstrates that it is not a negligible cost and should be taken seriously not only for its theoretical interest, but also for its practical importance. Interestingly, it offers an alternative explanation for why car buyers obtain significantly more of the surplus available under customer rebates than under dealer discounts, a finding that is counter to the simple invariance of incidence analogy (Busse, Silva-risso, and Zettelmeyer

[26] Computing ratios of consumer surplus instead does not make much sense, as v is arbitrary in our model.

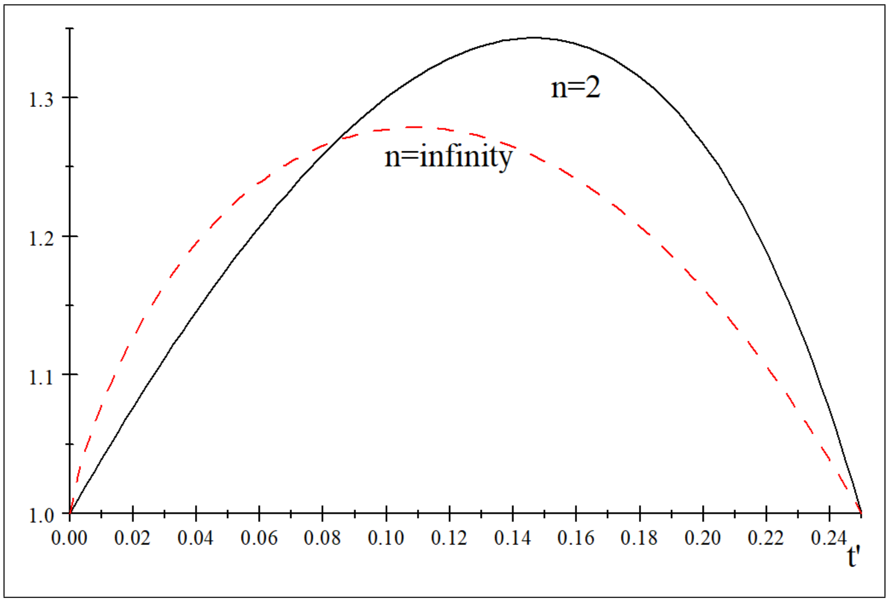

Figure 5: $\varphi(0,t)/c_E$ (the lower envelope of the dashed and the solid curve) as a function of $t' = t/(c_h - c_l)$, for $(q = 1/2)$.

2006). Busse et al. (2006) test several hypotheses and find evidence consistent with the asymmetric information hypothesis, that is, car buyers are disadvantaged in negotiations because they are less informed than dealers about the availability of dealer discounts. In contrast, the parties are symmetrically informed about the availability of customer rebates, which are always publicized to potential customers, often in prime-time television advertisements. Note that their explanation is based on the assumption that the information about dealer discounts is readily accessible to dealers. However, these discounts are often in the form of conditional discounts, depending on the geographical location and/or the specific equipment package, or "trim level". This means that there may be higher information costs for dealer discounts than for customer rebates.[27] Thus, the result that dealer discounts have a smaller pass-through can also be predicted by our model.

[27] According to a website specialized in automobile markets: "Even if you are the only customer in the dealership, there is still no guarantee you'll be able to get a deal offer in a flash. If you're taking out a loan, the sales manager might have to run your credit to get your credit score. He'll call the finance department to get your interest rate, and then look up specials and incentives on your car to make sure you're getting the right program offer for the right car. Sometimes it just takes a while to get all the information together." Matt Jones, "Behind the Scenes at A Car Dealership", April 29th, 2016, `https://www.edmunds.com/car-buying/behind-the-scenes-at-a-car-dealership.html`.

5 Conclusion

When consumers search, they incur costs. To provide consumers with the information they search for, sellers may also incur costs. This paper departs from most existing literature by assuming that sellers must make an effort to learn the cost of providing the goods/services before they bid against other sellers. Recent empirical studies have documented surprisingly few searches conducted by consumers when shopping for financial products. The lack of consumer search has been attributed to high search costs and non-price preferences. Our result, however, suggests that the choice of a small sample size when consumers search is not necessarily due to high search costs. It is also consistent with the existence of information costs. Empirical studies that do not take into account sellers' information costs may overestimate consumer search costs or the impact of other factors.

Clearly, a key assumption underlying our analysis is that the number of searches is publicly observable.[28] More realistically, the number of searches may be observed by some sellers but not all. Or, some measure of search intensity is observable, but the precise number of searches is not. In other words, the number of searches may be observed imperfectly. We hope to relax the assumption of perfect observability and explore related issues in future research.

Another potential extension of the model is to study search intermediaries such as brokers, who play a prominent role in relevant markets. In a two-sided matching model, Shi and Siow (2014) find that brokers can help reduce the costs of market participants through inventory management, thereby improving welfare. Similarly, introducing intermediaries into our model and analyzing the welfare consequence will help us better understand their roles in search markets.

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[28] Our paper also assumes that the cost realizations of sellers are identical. As a result, information acquisition is a waste. The result will undoubtedly be different if the sellers do not have the same production costs. In such a case, searches can also lead to cost discoveries, so the number of optimal searches will be more significant. We are grateful to a referee who made this point.

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Appendix

Proof of Lemma 1. For any $n \geq 2$, if $\alpha(n)$ were equal to 1 then, by the standard Bertrand/common value auction reasoning, sellers would have to set their price equal to the realized cost in equilibrium. As a result, net of the information cost t, their profits would be negative, and each seller could profitably deviate to not learning c and quoting price c_h (which yields a net profit of zero). Conversely, if $\alpha(n)$ were equal to 0, sellers would necessarily have to set price c_E in equilibrium, and thus earn zero profits. But, for each seller it would then be profitable to learn c, and to charge a price just below c_E if $c = c_l$ and a price equal to c_h if $c = c_h$. Indeed, the resulting change in profit is (arbitrarily close to)

$$q\frac{n-1}{n}(c_E - c_l) + (1-q)\frac{1}{n}(c_h - c_E) - t = q(1-q)(c_h - c_l) - t > 0.$$

This concludes the proof. ■

Proof of Lemma 2. The main tasks are showing *uniqueness* and deriving the explicit form of sellers' equilibrium strategy from necessary conditions. It will then be straightforward to verify that $n \mapsto (\alpha^*(n), F_l^*(\cdot|n), F_h^*(\cdot|n), F_b^*(\cdot|n))$ is indeed an equilibrium strategy.

Let $n \mapsto (\alpha(n), F_l(\cdot|n), F_h(\cdot|n), F_b(\cdot|n))$ be sellers' strategy in some symmetric SE. We will show that for all $n \geq 2$, $(\alpha(n), F_l(\cdot|n), F_h(\cdot|n), F_b(\cdot|n))$ must be $(\alpha^*(n), F_l^*(\cdot|n), F_h^*(\cdot|n), F_b^*(\cdot|n))$. To ease notation, we fix $n \geq 2$ and write (α, F_l, F_h, F_b) for $(\alpha(n), F_l(\cdot|n), F_h(\cdot|n), F_b(\cdot|n))$ and $(\alpha^*, F_l^*, F_h^*, F_b^*)$ for $(\alpha^*(n), F_l^*(\cdot|n), F_h^*(\cdot|n), F_b^*(\cdot|n))$. For $i \in \{l, h, b\}$, we set $\underline{p}_i := \min \operatorname{supp} F_i$ (the lowest price in the support of F_i) and $\bar{p}_i := \max \operatorname{supp} F_i$ (the highest price in the support of F_i). Moreover, Λ_i denotes the conditional expected profit (without substracting the information cost t) of an informed seller in state $i \in \{l, h\}$ (henceforth, a "type i seller") and Λ_b denote the conditional expected profit of an uninformed seller (henceforth, a "type b seller"), in the considered equilibrium (i.e., if all sellers play the mixed action (α, F_l, F_h, F_b) after having observed n). Of course, by definition of a mixed-strategy equilibrium, for each $i \in \{l, h, b\}$, the type i seller's expected profit from charging any price in supp F_i must be equal to Λ_i, which must be weakly higher than the expected profit associated with any price not in supp F_i. Furthermore, since $\alpha \in (0, 1)$ by Lemma 1, sellers must be indifferent between acquiring information and not acquiring information, that is:

$$(5.1) \qquad q\Lambda_l + (1-q)\Lambda_h - t = \Lambda_b.$$

We note first that neither F_h nor F_b can have an atom at a price $p > c_h$ (otherwise, charging a price just below the atom would increase profits) and that, similarly, F_l cannot have an atom at a price $p > c_l$. We now show $F_h = F_h^*$. First, $\underline{p_h} \geq c_h$ (otherwise, as α and hence the probability that all other sellers are also informed is strictly positive, charging price $\underline{p_h}$ would yield a negative expected profit, whereas charging price c_h guarantees a non-negative profit). Second, $\bar{p}_h \leq c_h$. Indeed, for the sake of deriving a contradiction, suppose that $\bar{p}_h > c_h$. It then follows that $\max\{\bar{p}_l, \bar{p}_b\} < \bar{p}_h$: otherwise $\bar{p}_i = \max\{\bar{p}_l, \bar{p}_b, \bar{p}_h\}$ for either $i = l$ or $i = b$. The "type" i for which this is true would then earn zero expected profit from charging the price $\bar{p}_i \in \text{supp } F_i$ (as the probability that the consumer accepts this offer is zero, given that F_l, F_b and F_h do not have atoms at prices above c_h), but strictly positive expected profits from certain lower prices (e.g. for $p = c_h$). Thus, $\max\{\bar{p}_l, \bar{p}_b\} < \bar{p}_h$, which (by an analogous reasoning) implies that the expected profit from charging price $\bar{p}_h$, and hence Λ_h, must be equal to 0. However, given that $\bar{p}_h > c_h$, a seller who has learned that the cost is c_h and charges a price in $(c_h, \bar{p}_h)$ earns a strictly positive expected profit (as $\alpha > 0$). This contradiction shows that we must have $\bar{p}_h \leq c_h$, and hence $F_h = F_h^*$. In particular, $\Lambda_h = 0$.

Given $F_h = F_h^*$, an argument akin to the one given in the previous paragraph shows $\max\{\bar{p}_l, \bar{p}_b\} \leq c_h$. Next, as we have argued above, F_l cannot have atoms at prices above c_l. Moreover, $\Lambda_b \geq 0$, as an uninformed buyer can guarantee a zero expected profit by charging a sufficiently high price (which is equivalent to not making an offer). Thus, by (5.1) and $\Lambda_h = 0$, we also have $\Lambda_l = \frac{t+\Lambda_b}{q} > 0$, and hence $\underline{p}_l > c_l$. In particular, F_l is atomless.

We now turn to a key step in the proof, which is showing that F_b has no atoms. First, F_b cannot have an atom at a price strictly above c_h (see above), a price weakly below c_l (as this would entail a negative expected profit) or at c_h, as deviating from c_h to a slightly lower price would then increase the probability of gaining $c_h - c_l$ (in state l) by a discrete amount (as the probability that all other sellers are uninformed is strictly positive) but lead only to a marginal loss in state h.

Assume then that F_b has an atom at a price $p_m \in (c_l, c_h)$. Let $P^w(p|c_i)$ denote the probability of winning with bid p in state $i \in \{l, h\}$ (i.e., conditional on the state), provided that all other $n-1$ sellers behave according to (α, F_l, F_h, F_b). We also define $P^w(p^-|c_i) := \lim_{p' \to p, p' < p} P^w(p'|c_i)$, $P^w(p^+|c_i) := \lim_{p' \to p, p' > p} P^w(p'|c_i)$, as well as $\Delta_i^- := P^w(p_m^-|c_i) - P^w(p_m|c_i)$ and $\Delta_i^+ := P^w(p_m|c_i) - P^w(p_m^+|c_i)$. Thus, for a type b seller (who believes that all others behave according to (α, F_l, F_h, F_b)), the (discrete) effect on his expected utility of marginally decreasing his price at p_m is $q(p_m - c_l)\Delta_l^- + (1-q)(p_m - c_h)\Delta_h^-$, and the (discrete) effect of marginally increasing his price is $q(p_m - c_l)(-\Delta_l^+) + (1-q)(p_m - c_h)(-\Delta_h^+)$. A necessary condition for equilibrium is that both of these effects are weakly negative, i.e.,

$$q(p_m - c_l)\Delta_l^- + (1-q)(p_m - c_h)\Delta_h^- \leq 0 \tag{5.2}$$

$$q(p_m - c_l)\Delta_l^+ + (1-q)(p_m - c_h)\Delta_h^+ \geq 0. \tag{5.3}$$

In particular, as $q(p_m - c_l) > 0$ and $(1-q)(p_m - c_h) < 0$, it follows that we must have

$$\frac{\Delta_h^-}{\Delta_l^-} \geq \frac{\Delta_h^+}{\Delta_l^+} \Leftrightarrow \frac{\Delta_h^-}{\Delta_h^+} \geq \frac{\Delta_l^-}{\Delta_l^+}, \tag{5.4}$$

and that if equality holds in (5.4), equality must hold in both (5.2) and (5.3).

We now spell out Δ_l^-, Δ_l^+, Δ_h^- and Δ_h^+. To this end, let $P_{(k,j,i)}$ denote the probability, conditional on state $i \in \{l, h\}$, that exactly $k \in \{1, ..., n-1\}$ other sellers are uninformed, exactly $j \in \{1, ..., k\}$ of these uninformed sellers bid exactly p_m, the remaining $k - j$ uninformed sellers make bids above p_m, and all $n - 1 - k$ informed sellers also bid higher than p_m. Setting $\nu := F_b(p_m) - \lim_{p \to p_m, p < p_m} F_b(p)$ (the mass of the atom), we can express $P_{(k,j,l)}$ and $P_{(k,j,h)}$ as follows (note that $1 - F_h(p_m) = 1$, as $p_m < c_h$, and that, by the usual convention, $0^0 = 1$).

$$P_{(k,j,l)} = \binom{n-1}{k} (\alpha(1 - F_l(p_m)))^{n-1-k} (1-\alpha)^k \binom{k}{j} \nu^j (1 - F_b(p_m))^{k-j} \tag{5.5}$$

$$P_{(k,j,h)} = \binom{n-1}{k} \alpha^{n-1-k} (1-\alpha)^k \binom{k}{j} \nu^j (1 - F_b(p_m))^{k-j} \tag{5.6}$$

In particular, $P_{(k,j,l)} = P_{(k,j,h)}(1 - F_l(p_m))^{n-1-k}$. We have (recall that if a seller is one of $j+1$ sellers making the lowest offer, the consumer buys from him with probability $1/(j+1)$):

$$\Delta_l^- = \sum_{k=1}^{n-1}\sum_{j=1}^{k} P_{(k,j,l)}\frac{j}{j+1} = \sum_{j=1}^{n-1}\frac{j}{j+1}\sum_{k=j}^{n-1} P_{(k,j,h)}(1 - F_l(p_m))^{n-1-k} \tag{5.7}$$

$$\Delta_l^+ = \sum_{k=1}^{n-1}\sum_{j=1}^{k} P_{(k,j,l)}\frac{1}{j+1} = \sum_{j=1}^{n-1}\frac{1}{j+1}\sum_{k=j}^{n-1} P_{(k,j,h)}(1 - F_l(p_m))^{n-1-k} \tag{5.8}$$

$$\Delta_h^- = \sum_{k=1}^{n-1}\sum_{j=1}^{k} P_{(k,j,h)}\frac{j}{j+1} = \sum_{j=1}^{n-1}\frac{j}{j+1}\sum_{k=j}^{n-1} P_{(k,j,h)} \tag{5.9}$$

$$\Delta_h^+ = \sum_{k=1}^{n-1}\sum_{j=1}^{k} P_{(k,j,h)}\frac{1}{j+1} = \sum_{j=1}^{n-1}\frac{1}{j+1}\sum_{k=j}^{n-1} P_{(k,j,h)} \tag{5.10}$$

Note also that an atom of F_b cannot belong to suppF_l: otherwise the type l seller would like to set prices arbitrarily close to but below the atom, contradicting the existence of a best response.

We now proceed to show that our assumption that F_b has an atom $p_m \in (c_l, c_h)$ leads to a contradiction.

Assume first that $p_m < \underline{p_l}$. This yields $1 - F_l(p_m) = 1$, and hence $\Delta_l^- = \Delta_h^-$ and $\Delta_l^+ = \Delta_h^+$. Thus, (5.4) holds with equality, so equality holds in (5.2) and (5.3), and the location of the atom is *uniquely pinned down* by $q(p_m - c_l) + (1-q)(p_m - c_h) = 0$, i.e., $p_m = c_E$. Moreover, *because* (5.3) holds with equality (i.e., for an uninformed seller, the limit of expected profits associated with prices converging from above to p_m is equal to (not strictly lower than) the expected profit associated with setting price p_m), there can be no gap in supp $F_l \cup$ supp F_b with lower boundary p_m: otherwise, the uninformed seller would get a strictly higher expected profit from charging any price in this gap than from charging p_m (as different prices in the gap entail the exact same winning probability). By analogous reasoning, all prices in the non-empty interval $(p_m, \underline{p_l}]$ belong to supp F_b and hence yield the same expected profit Λ_b for the uninformed seller, i.e., $q(p - c_l)[\alpha + (1-\alpha)(1 - F_b(p))]^{n-1} + (1-q)(p - c_h)[\alpha + (1-\alpha)(1 - F_b(p))]^{n-1} = \Lambda_b$ for all $p \in (p_m, \underline{p_l}]$. As $(1-q)(p - c_h)[\alpha + (1-\alpha)(1 - F_b(p))]^{n-1}$ is increasing in p on $(p_m, \underline{p_l}]$ (as $\underline{p_l} < c_h$), this implies that the expected profit for a type l seller from charging such prices, $(p - c_l)[\alpha + (1-\alpha)(1 - F_b(p))]^{n-1}$ is decreasing in p over this interval. This contradicts $\underline{p_l} \in$ supp F_l.

Assume thus from now on that $p_m > \underline{p_l}$. Consider the case $n = 2$ first. Then, $\Delta_l^- = \Delta_l^+$ and $\Delta_h^- = \Delta_h^+$, which again implies that (5.4) holds with equality, so equality holds in (5.2) and (5.3), and the location of the atom is uniquely pinned down at $p_m = c_E$. An argument analogous to the one given in the previous paragraph then shows that $p_m < \bar{p}_l$ is not possible (the role played by $\underline{p_l}$ in the previous paragraph is now played by the smallest price in supp F_l that lies above p_m). Thus, we must have $p_m > \bar{p}_l$. But, because of the equality in (5.3), it again follows that there can be no gap in supp F_b starting at p_m, and that the entire interval $[p_m, c_h]$ must then belong to supp F_b. In particular, $\Lambda_b = 0$, and $q(p-c_l)((1-\alpha)(1-F_b(p)))^{n-1} + (1-q)(p-c_h)((\alpha+(1-\alpha)(1-F_b(p)))^{n-1} = \Lambda_b = 0$ for all $p \in (p_m, c_h]$. This implies

$$F_b(p) = 1 - \frac{\alpha}{1-\alpha}\left(\left(\frac{q(p-c_l)}{(1-q)(c_h-p)}\right)^{\frac{1}{n-1}} - 1\right)^{-1} \tag{5.11}$$

for all $p \in (p_m, c_h]$. In particular, p_m would have to be greater than the price for which the right hand side of (5.11) is 0, which is false, because the latter price is $p = \frac{(1-q)c_h + q(1-\alpha)^{n-1}c_l}{(1-q)+q(1-\alpha)^{n-1}} > c_E$ (where the inequality holds due to $\alpha > 0$). Contradiction.

Consider now the remaining case, $n \geq 3$ and $p_m > \underline{p_l}$ (i.e., $1 - F_l(p_m) < 1$). In this case, $(1-F_l(p_m))^{n-1-k}$ is non-constant and nondecreasing in $k \in \{1, ..., n-1\}$ (it is strictly increasing if $1-F_l(p_m) > 0$; if $1-F_l(p_m) = 0$, then $(1-F_l(p_m))^{n-1-k} = 0$ for all $k \leq n-2$ and $(1-F_l(p_m))^{n-1-k} = 1$ for $k = n-1$). This observation and formulas (5.7)-(5.10) imply however that $\frac{\Delta_h^-}{\Delta_h^+} < \frac{\Delta_l^-}{\Delta_l^+}$, i.e., (5.4) does not hold. Indeed, note that $\frac{\Delta_l^-}{\Delta_l^+}$ can be interpreted as the expected value of the probability distribution on $\{1, ..., n-1\}$ that puts mass $\frac{1}{j+1}\sum_{k=j}^{n-1} P_{(k,j,h)}(1-F_l(p_m))^{n-1-k}/\Delta_l^+$ on point $j \in \{1, ..., n-1\}$, and that $\frac{\Delta_h^-}{\Delta_h^+}$ can similarly be interpreted as the expectation of the probability distribution on $\{1, ..., n-1\}$ that puts mass $\frac{1}{j+1}\sum_{k=j}^{n-1} P_{(k,j,h)}/\Delta_h^+$ on point j. It is straightforward to check that because $(1-F_l(p_m))^{n-1-k}$ is non-constant and nondecreasing in k, the former distribution first order stochastically dominates the latter, which implies $\frac{\Delta_l^-}{\Delta_l^+} > \frac{\Delta_h^-}{\Delta_h^+}$. This concludes the proof that F_b is atomless.

It now follows easily that (α, F_l, F_h, F_b) must be equal to $(\alpha^*, F_l^*, F_h^*, F_b^*)$. First, we have already seen that $\max\{\bar{p}_l, \bar{p}_b\} \leq c_h$. Secondly, as F_b and F_l are atomless, supp $F_l \cup$ supp F_b must be equal to $[\min\{\underline{p_l}, \underline{p_b}\}, c_h]$ (if supp $F_l \cup$ supp $F_b \cup$ supp F_h^* had a gap, the type of seller whose bid distribution contains the lower boundary of the gap could increase his profit by choosing a price in the gap instead). Third, we must have $\bar{p}_l = \underline{p_b}$. Indeed, consider two prices $p_1 < p_2$ from supp F_l. The indifference condition for the type l seller yields $(p_1 - c_l)(\alpha(1 - F_l(p_1)) + (1 - \alpha)(1 - F_b(p_1)))^{n-1} = (p_2 - c_l)(\alpha(1 - F_l(p_2)) + (1 - \alpha)(1 - F_b(p_2)))^{n-1}$. As the expected profit for the type b seller from charging price $p \in \{p_1, p_2\}$ is $q(p - c_l)(\alpha(1 - F_l(p)) + (1 - \alpha)(1 - F_b(p)))^{n-1} + (1 - q)(p - c_h)(\alpha + (1 - \alpha)(1 - F_b(p)))^{n-1}$ and $(p - c_h)(\alpha + (1 - \alpha)(1 - F_b(p)))^{n-1}$ is strictly increasing in p for $p < c_h$, the type b seller strictly prefers p_2 to p_1. Thus, $\bar{p}_l = \underline{p_b}$. Fourth, it now follows that F_b must be of the form (5.11) with $\underline{p_b} = \frac{(1-q)c_h + q(1-\alpha)^{n-1}c_l}{(1-q)+q(1-\alpha)^{n-1}}$ (see the argument in the paragraph where (5.11) was derived) and $\bar{p}_b = c_h$. Fifth, the indifference condition for type l sellers, i.e., that $(p - c_l)(\alpha(1 - F_l(p)) + (1 - \alpha))^{n-1} = \Lambda_l = t/q$ must hold for $p \in [\underline{p_l}, \bar{p}_l]$ yields $F_l(p) = \left(1 - \left(\frac{t}{q(p-c_l)}\right)^{1/(n-1)}\right)/\alpha$, as well $\underline{p_l} = c_l + t/q$ and $\bar{p}_l = c_l + \frac{t}{q(1-\alpha)^{n-1}}$. From $\bar{p}_l = \underline{p_b}$ it then follows that $\alpha = 1 - \left(\frac{t(1-q)}{q((c_h-c_l)(1-q)-t)}\right)^{1/(n-1)}$. Plugging this back into the formula for $\underline{p_b}$, we obtain $\underline{p_b} = c_h - t/(1 - q)$. This concludes the proof that (α, F_l, F_h, F_b)=$(\alpha^*, F_l^*, F_h^*, F_b^*)$.

That $(\alpha^*, F_l^*, F_h^*, F_b^*)$ indeed satisfies the necessary and sufficient conditions for a symmetric mixed-strategy equilibrium is almost immediate from the above derivations, so we omit repeating the details. Finally, as for each $n \geq 2$, sellers earn zero expected net profits $(0 = \Lambda_b = q\Lambda_l + (1 - q)\Lambda_h - t)$ and the total surplus equals $v - c_E - n(s + \alpha^*(n)t)$, it follows that the expected price paid is $c_E + n\alpha^*(n)t$. ■

Proof of Lemma 3. By Lemma 2, $\underline{p_l^*} = c_l + t/q$, $\bar{p}_l^* = \underline{p_b^*} = c_h - t/(1 - q)$, $\bar{p}_b^* = c_h$. The first two results immediately follow. Last, since α^* decreases with t, $F_b^*(p)$ increases with t, implying that blind bids for a smaller t first-order stochastically dominate blinds bids for a larger t. ■

To prepare the proof of Lemma 4 and Proposition 1, we define

$$g(\gamma) := e^{\gamma-2}(4/\gamma - 1) \quad \text{and} \quad h(\gamma) := e^{-\gamma}(2\gamma + 1)$$

for all $\gamma > 0$ and note the following observations.

Observation 2 *(i) g is strictly decreasing, $g(2) = 1$, $\lim_{\gamma\to 0} g(\gamma) = \infty$ and $\lim_{\gamma\to\infty} g(\gamma) = -\infty$.*

(ii) h is unimodal, attains its maximum at $\gamma = \frac{1}{2}$, and satisfies $h(1) = g(1)$, $h(\gamma) < g(\gamma)$ for all $\gamma \in [0, 2] \setminus \{1\}$, and $h(0) = 1$. Moreover, the function $h \circ \gamma$ is concave in the part where it is decreasing, i.e., for $t > \gamma^{-1}(\frac{1}{2}) = \frac{q(1-q)(c_h - c_l)}{(1-q)e^{1/2}+q}$.[29]

Observation 2 immediately implies the following facts.

Observation 3 *(i) For each $s \geq 0$, there is a unique value $t^*(s) \in (0, q(1-q)(c_h - c_l))$ for which $1 + s/t = g(\gamma(t))$ (hence, $1 + s/t \gtreqless g(\gamma(t))$ for $t^*(s) \gtreqless t$). Moreover, $\gamma(t^*(0)) = 2$, and $\gamma(t^*(s))$ is strictly decreasing in s, i.e., $t^*(s)$ is strictly increasing in s.*

(ii) Let $\bar{s}$ denote the unique value of s for which $\min_{t \in (0, q(1-q)(c_h - c_l))} 1 + s/t - h(\gamma(t)) = 0$. For any $s < \bar{s}$ (i.e., $1 + s/t < h(\gamma(t))$ holds for some *value of t), there are unique values $t_1(s)$ and $t_2(s)$, satisfying $t^*(s) \leq t_1(s) < t_2(s) < q(1-q)(c_h - c_l)$ and $\gamma(t_2(s)) < 1/2$ such that $1 + s/t < h(\gamma(t))$ for $t \in (t_1(s), t_2(s))$ and $1 + s/t > h(\gamma(t))$ for $t < t_1(s)$ and for $t > t_2(s)$. Moreover, $t^*(s) = t_1(s)$ if and only if $\gamma(t^*(s)) = 1$, and $\gamma(t^*(s)) > 1$ implies $\gamma(t_1(s)) > 1$.*

Part (i) of Observation 3 is obvious. For part (ii), note that the strictly decreasing function $t \mapsto 1 + s/t$ can intersect $h \circ \gamma$ at most once in the interval where $h \circ \gamma$ is increasing, i.e., for $t \leq \gamma^{-1}(\frac{1}{2})$. Moreover, if $1 + s/t - h(\gamma(t)) < 0$ holds for some value of t, then, as $t \mapsto 1 + s/t$ is strictly convex, $h \circ \gamma$ is strictly concave for $t > \gamma^{-1}(\frac{1}{2})$, and $h(0) = 1 < 1 + \frac{s}{q(1-q)(c_h - c_l)}$, there must be exactly two points of intersection, $t_1(s) < t_2(s)$ (and $t_2(s)$ must be in the part where $h \circ \gamma$ is decreasing). Next, if $\gamma(t^*(s)) \neq 1$ then $h(\gamma(t^*(s))) < g(\gamma(t^*(s)))$, so that $1 + s/t > h(\gamma(t))$ for t close to $t^*(s)$, which implies $t_1(s) > t^*(s)$. On the other hand, if $\gamma(t^*(s)) = 1$ then (as $h(1) = g(1)$), $t_1(s) = t^*(s)$. Finally, if $\gamma(t^*(s)) > 1$ then $h(1) = g(1)$ and the monotonicity properties of h imply $\gamma(t_1(s)) > 1$. This explains Observation 3.

Lemma 4 *(i) If $t \leq t^*(s)$ (i.e., $1 + s/t \geq g(\gamma(t))$) then $\psi(n, s, t)$ strictly increases with n on $(1, \infty)$.*

(ii) If $t > t^*(s)$ it holds:

(ii.a) If $s = 0$, then $\psi(n, s, t)$ is unimodal in n on $(1, \infty)$.

[29] The last claim is the only part of Observation 2 that is not completely straightforward to check, so let us explain how to prove it. As $(h \circ \gamma)''(t) = h''(\gamma(t))(\gamma'(t))^2 + h'(\gamma(t))\gamma''(t)$, we compute $h'(\gamma) = e^{-\gamma}(1-2\gamma)$, $h''(\gamma) = e^{-\gamma}(2\gamma - 3)$, $\gamma'(t) = -\frac{a}{t(a-t)}$ and $\gamma''(t) = \frac{a(a-2t)}{t^2(a-t)^2}$, where $a = (1-q)(c_h - c_l)$. Clearly, $h''(\gamma) < 0$ and $|h''(\gamma)| > h'(\gamma)$ for all $\gamma < 1/2$. As $a > q(1-q)(c_h - c_l) > t$, we also have $\gamma'(t)^2 > |\gamma''(t)|$, and it follows that $(h \circ \gamma)''(t) < 0$ for all $t > \frac{q(1-q)(c_h - c_l)}{(1-q)e^{1/2}+q}$.

(ii.b) If $s > 0$, then $\psi(n,s,t)$ has exactly two critical points on $(1,\infty)$, a local maximum $n_1 = n_1(s,t)$ and a local minimum $n_2 = n_2(s,t)$, satisfying $n_1 < n_2$. If $1 + s/t < h(\gamma(t))$ then $n_1 < 2 < n_2$. If $1 + s/t > h(\gamma(t))$ and $\gamma(t) > 1$ then $2 < n_1 < n_2$. If $1 + s/t > h(\gamma(t))$ and $\gamma(t) < 1$ then $n_1 < n_2 < 2$.

(iii) For all (s,t), $\lim_{n\to\infty} \psi(n,s,t) - (ns + \gamma(t)t) = 0$.

Proof of Lemma 4. To prove the lemma, it will be useful to define the following function for all $z > 0$, $\gamma > 0$ and $x \geq 0$:

$$\phi(z,\gamma,x) := \gamma\frac{(1+x)e^z - 1}{z(z+\gamma)}. \tag{5.12}$$

Indeed, noting that $\alpha^*(n,t) = 1 - e^{-\gamma(t)/(n-1)}$ and defining $z(n,t) := \frac{\gamma(t)}{n-1}$, we have $\alpha^*(n,t) = 1 - e^{-z(n,t)}$ and $n = \frac{\gamma(t)}{z(n,t)} + 1$, and we may express the expected markup $\psi(n,s,t)$ as follows:

$$\psi(n,s,t) = (\alpha^*(n,t)t + s)n = \left(\left(1 - e^{-z(n,t)}\right)t + s\right)\left(\frac{\gamma(t)}{z(n,t)} + 1\right). \tag{5.13}$$

Using $\frac{\partial z}{\partial n}(n,t) = -\frac{z^2(n,t)}{\gamma(t)}$, a straightforward computation then yields

$$\frac{\partial \psi}{\partial n}(n,s,t) = tz(n,t)e^{-z(n,t)}\frac{\gamma(t) + z(n,t)}{\gamma(t)}(\phi(z(n,t),\gamma(t),s/t) - 1). \tag{5.14}$$

Thus, $\frac{\partial\psi}{\partial n}(n,s,t)$ has the same sign as $\phi(z(n,t),\gamma(t),s/t) - 1$.

We note the following properties of the function ϕ (defined on $\mathbb{R}^2_{++} \times \mathbb{R}_+$). First, ϕ is strictly increasing in γ (as $\gamma/(z+\gamma)$ is). Second, ϕ is strictly convex in z. Indeed, $\frac{\partial^2}{\partial z^2}\frac{e^z - 1}{z(z+\gamma)} = \frac{a\gamma^2 + b\gamma + c}{z^3(z+\gamma)^3}$, where $a = 2e^z + z^2e^z - 2ze^z - 2$, $b = 2z(3e^z + z^2e^z - 3ze^z - 3)$ and $c = z^2(6e^z + z^2e^z - 4ze^z - 6)$, and it can be verified numerically that $a > 0$, $c > 0$ and $b^2 - 4ac < 0$ for all $z > 0$, which implies that $a\gamma^2 + b\gamma + c > 0$ and thus $\frac{\partial^2}{\partial z^2}\frac{e^z-1}{z(z+\gamma)} > 0$. Thus, for $x = 0$, ϕ is strictly convex in z. Moreover, as the sum of two strictly convex functions is strictly convex, it follows that the functions $\frac{e^z}{z(z+\gamma)}$ (as the sum of $\frac{e^z-1}{z(z+\gamma)}$ and the strictly convex function $\frac{1}{z(z+\gamma)}$) and then also ϕ, for arbitrary $x > 0$, are strictly convex in z. Third, for any $\gamma > 0$, we have

$$\lim_{z\to\infty} \phi(z,\gamma,x) = +\infty \quad \text{for all} \quad x \geq 0, \tag{5.15}$$

$$\lim_{z\to 0} \phi(z,\gamma,x) = \begin{cases} +\infty & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{5.16}$$

Indeed, (5.15) is immediate from the behavior of the exponential function for $z \to \infty$, (5.16) in the case $x > 0$ is immediate as the denominator of ϕ tends to 0 whereas the numerator tends to $\gamma x > 0$, and (5.16) for $x = 0$ follows easily from the Taylor series representation of $e^z - 1$.

The above observations imply that for each $x > 0$ (and each $\gamma > 0$), $\phi(\cdot,\gamma,x)$ is strictly decreasing up to its global minimum, determined by the first order condition

$$\phi_z(z,\gamma,x) = \gamma\frac{(\gamma+2z) + e^z(-2z-\gamma+z\gamma+z^2)(1+x)}{z^2(z+\gamma)^2} = 0, \tag{5.17}$$

and strictly increasing from thereon. For the case $x = 0$, a straightforward application of L'Hôpital's rule (Estrada and Pavlovic 2017) yields $\lim_{z\to 0}\phi_z(z,\gamma,0) = \frac{\gamma-2}{2\gamma}$. Combined with (5.16), it follows that $\phi(\cdot,\gamma,0)$ has its global minimum at the solution of (5.17) and attains a value below 1 at that minimum if $\gamma < 2$, whereas, if $\gamma \geq 2$, $\phi(\cdot,\gamma,0) - 1$ is positive on $(0,\infty)$. Hence, if $\gamma(t) \geq 2$, i.e., for $t \leq t^*(0)$, $\psi(\cdot,0,t)$ is strictly increasing on $(1,\infty)$, whereas $\psi(\cdot,0,t)$ is unimodal if $\gamma(t) < 2$, i.e., for $t > t^*(0)$: in the latter case, the equation $\phi(z,\gamma(t),0) = 1$ has a unique solution $\hat{z} \in (0,\infty)$, and $\psi(\cdot,0,t)$ is strictly increasing for values of n satisfying $z(n,t) = \gamma(t)/(n-1) > \hat{z}$ and strictly decreasing for values of n for which $z(n,t) < \hat{z}$. We have thus proven part (ii.a) of the lemma, as well as part (i) in the case $s = 0$.

Next, for any $x > 0$, the unique solution $z^* = z^*(\gamma,x)$ of (5.17) solves

$$(1+x)e^z = \frac{\gamma+2z}{\gamma+2z-z\gamma-z^2}, \tag{5.18}$$

which yields $\min_z \phi(z,\gamma,x) = \phi(z^*,\gamma,x) = \gamma\frac{(1+x)e^{z^*}-1}{z^*(z^*+\gamma)} = \frac{\gamma}{\gamma+2z^*-z^*\gamma-z^{*2}}$. Recalling that $g(\gamma) = e^{\gamma-2}(4/\gamma-1)$ is strictly decreasing in γ, with $\lim_{\gamma\to 0} g(\gamma) = \infty$ and $\lim_{\gamma\to\infty} g(\gamma) = -\infty$, we let $\gamma^* = \gamma^*(x)$ denote the unique value of γ solving $1 + x = g(\gamma)$ and note that $\gamma^*(x)$ is strictly decreasing in x with $\lim_{x\to 0}\gamma^*(x) = 2$. Moreover, for $\gamma = \gamma^*$, $z = 2-\gamma$ solves (5.18), i.e., we have $z^* = 2-\gamma^*$, and consequently also $\phi(z^*,\gamma^*,x) = \frac{\gamma^*}{\gamma^*+2z^*-z^*\gamma-z^{*2}} = 1$ in this case. Since ϕ_γ is strictly positive, it follows that $\min_z \phi(z,\gamma,x) \geq 1$ for $\gamma \geq \gamma^*$ and $\min_z \phi(z,\gamma,x) < 1$ for $\gamma < \gamma^*$.

In particular, for $t \leq t^*(s)$, which is equivalent to $\gamma(t) \geq \gamma^*(s/t^*(s))$, as well as to $\gamma(t) \geq \gamma^*(s/t)$, or $1+s/t \geq g(\gamma(t))$, ψ is strictly increasing in n. This concludes the proof of (i).

Hence, we assume from now on that $s > 0$ and $t > t^*(s)$ (i.e., $1+s/t < g(\gamma(t))$, or $\gamma(t) < \gamma^*(s/t)$). From the above arguments, it follows that $\min_z \phi(z,\gamma(t),s/t) < 1$ in this case, so that the equation $\phi(z,\gamma(t),s/t) = 1$ has two solutions on $(0,\infty)$. Denote them by $z_1 = z_1(s,t)$ and $z_2 = z_2(s,t)$, with $z_1 > z_2$. It follows that $\psi(\cdot,s,t)$ has exactly two critical points on $(1,\infty)$, a local maximum at $n_1 = \gamma(t)/z_1 + 1$ and local minimum at $n_2 = \gamma(t)/z_2 + 1 > n_1$.

We now identify how the critical points n_1 and n_2 are located relative to $n = 2$ (where $z(n,t) = \gamma(t)$). To this end, note that

$$\phi(\gamma,\gamma,x) = \frac{(1+x)e^\gamma - 1}{2\gamma} \gtrless 1 \text{ if and only if } 1+x \gtrless h(\gamma).$$

It follows that if $1+s/t < h(\gamma(t))$, i.e., for $t \in (t_1(s),t_2(s))$ (see Observation 3), $\phi(\gamma(t),\gamma(t),s/t) < 1$, which implies $z_2 < \gamma(t) < z_1$, i.e., $n_1 < 2 < n_2$. We thus consider the case $1+s/t > h(\gamma(t))$, i.e., $t \in (t^*(s),t_1(s)) \cup (t_2(s),q(1-q)(c_h-c_l))$, from now on (the first interval is non-empty if $\gamma(t^*(s)) \neq 1$, see Observation 3). Hence, $\phi(\gamma(t),\gamma(t),s/t)) > 1$, which implies in particular $\phi_z(\gamma(t),\gamma(t),s/t) = \frac{3+e^{\gamma(t)}(2\gamma(t)-3)(1+s/t)}{4\gamma(t)^2} \neq 0$ (as $\min_z \phi(z,\gamma(t),s/t) < 1$, $\gamma(t)$ is not the minimum of $\phi(\cdot,\gamma(t),s/t)$). As $\phi_z(\gamma(t_2(s)),\gamma(t_2(s)),s/t_2(s)) \neq 0$ ($\phi(\gamma(t_2(s)),\gamma(t_2(s)),s/t_2(s)) = 1$, so $z = \gamma(t_2(s))$ is not the minimum of $\phi(\cdot,\gamma(t_2(s)),s/t_2(s)))$, $\phi_z(\gamma(\cdot),\gamma(\cdot),s/\cdot)$ is continuous, and $(t_2(s),q(1-q)(c_h-c_l))$ is connected, the intermediate value theorem implies $\text{sign}(\phi_z(\gamma(t),\gamma(t),s/t)) = \text{sign}(\phi_z(\gamma(t_2(s)),\gamma(t_2(s)),s/t_2(s)))$ for all $t \in (t_2(s),q(1-q)(c_h-c_l))$. Using $1+s/t_2(s) = h(\gamma(t_2(s)))$, it follows that

$$3+e^{\gamma(t_2(s))}(2\gamma(t_2(s))-3)(1+s/t_2(s)) = 3+(2\gamma(t_2(s))-3)(2\gamma(t_2(s))+1) = 4\gamma(t_2(s))(\gamma(t_2(s))-1) < 0,$$

where the last inequality uses $\gamma(t_2(s)) < \frac{1}{2}$ (see Observation 3). Thus, for all $t \in (t_2(s),q(1-q)(c_h-c_l))$ (for which $\gamma(t) < \gamma(t_2(s)) < \frac{1}{2}$), $z = \gamma(t)$ is in the decreasing part of the convex function $\phi(\cdot,\gamma(t),s/t)$, and hence $\gamma(t) < z_2 < z_1$, i.e., $n_1 < n_2 < 2$. Similarly, $\text{sign}(\phi_z(\gamma(t),\gamma(t),s/t)) =$

$\mathrm{sign}(\phi_z(\gamma(t_1(s)),\gamma(t_1(s)),s/t_1(s))) = \mathrm{sign}(\gamma(t_1(s))-1)$ for all $t \in (t^*(s), t_1(s))$. If $\gamma(t^*(s)) < 1$ then $\gamma(t) < 1$ for all $t \in (t^*(s), t_1(s))$, whereas if $\gamma(t^*(s)) > 1$ then (see Observation 3) $\gamma(t) > 1$ for all $t \in (t^*(s), t_1(s))$. In the latter case, $z = \gamma(t)$ is in the increasing part of the convex function $\phi(\cdot, \gamma(t), s/t)$, and hence $z_2 < z_1 < \gamma(t)$, i.e., $2 < n_1 < n_2$. This concludes the proof of $(ii.b)$.

Result (iii) follows immediately from formula (5.13) and $1 - e^{-z} = z + o(z)$ as $z \to 0$. ■

Proof of Proposition 1. If $s = 0$, by parts (i) and $(ii.a)$ of Lemma 4, we only need to compare $\psi(2, 0, t)$ and $\lim_{n\to\infty} \psi(n, 0, t)$ (for $t > t^*(0)$). By Lemma 4 (iii), $\lim_{n\to\infty} \psi(n, 0, t) = t\gamma(t)$, whereas $\psi(2, 0, t) = 2t(1 - e^{-\gamma(t)})$. Hence, $\psi(2, 0, t) \gtrless \lim_{n\to\infty} \psi(n, 0, t)$ for $\hat{\gamma} \gtrless \gamma(t)$, where $\hat{\gamma} \approx 1.594$ is the unique value of γ such that $1 - e^{-\gamma} = \gamma/2$. As $t = \frac{q(1-q)(c_h-c_l)}{q+(1-q)e^{\gamma(t)}}$ and $e^{1.594} \approx 4.92$, claim (i) follows.

If $0 < s < \bar{s}$, Lemma 4 (i) implies that $n^o = 2$ for $t \le t^*(s)$, and Lemma 4 (ii.b) shows that for $t \in (t_1(s), t_2(s))$ (i.e., $1 + s/t < h(\gamma(t))$), $n = 2$ lies between the local maximum n_1 and the local minimum n_2, which implies $n^o = n_2 > 2$. Also, for all $t \in (t_2(s), q(1-q)(c_h - c_l))$, we have $1 + s/t > h(\gamma(t))$ and $\gamma(t) < \gamma(t_2(s)) < 1/2$, so that, by Lemma 4 (ii.b), $n_2 < 2$ and thus $n^o = 2$.

Next, if $\gamma(t^*(s)) \le 1$, then $\gamma(t) < 1$ (and $1 + s/t > h(\gamma(t))$) for all $t \in (t^*(s), t_1(s))$, i.e., $n^o = 2$ in this case. This completes the proof of (iii).

To complete the proof of (ii), note that if $\gamma(t^*(s)) > 1$ then $\gamma(t_1(s)) > 1$, so that Lemma 4 (ii.b) implies $2 < n_1(s,t) < n_2(s,t)$ for all t from the non-empty interval $(t^*(s), t_1(s))$. Thus, for these values of t, we have to compare $\psi(2, s, t)$ and $\psi(n_2(s,t), s, t)$ to determine n^o. As $\psi(\cdot, s, t^*(s))$ is strictly increasing and $\frac{\partial}{\partial n}\psi(2, s, t^*(s)) > 0$ (recall from the proof of Lemma 4 that the only point where $\frac{\partial}{\partial n}\psi(\cdot, s, t^*(s))$ vanishes corresponds to $z = 2 - \gamma(t^*(s)) < 1 < \gamma(t^*(s))$, i.e., a value of n strictly above 2), it follows by continuity that $n^o = 2$ for t sufficiently close to $t^*(s)$. On the other hand, for $t = t_1(s)$, we have $2 = n_1(s,t) < n_2(s,t)$ (so $\psi(2, s, t_1(s)) > \psi(n_2(s, t_1(s)), s, t_1(s))$), and it follows by continuity that $n^o > 2$ for t sufficiently close to $t_1(s)$. With regard to Remark **??**, we also note that formula (5.13) implies $\psi(2, s, t) = 2((1-e^{-\gamma(t)})t+s) = 2t(1+s/t-e^{-\gamma(t)})$, and $\psi(n_2(s,t), s, t) = \left(\left(1 - e^{-z(n_2(s,t),t)}\right) t + s\right)\left(\frac{\gamma(t)}{z(n_2(s,t),t)} + 1\right) = te^{-z(n_2(s,t),t)}\frac{(\gamma(t)+z(n_2(s,t),t))^2}{\gamma(t)}\phi(z(n_2(s,t),t), \gamma(t), s/t) = te^{-z(n_2(s,t),t)}\frac{(\gamma(t)+z(n_2(s,t),t))^2}{\gamma(t)}$, where the final step uses $\phi(z(n_2(s,t),t), \gamma(t), s/t) = 1$ (recall (5.14) and that $n_2(s,t)$ is a critical point of $\psi(\cdot, s, t)$). Thus, $n^o > 2$ for those values $t \in (t^*(s), t_1(s))$ satisfying $e^{-z(n_2(s,t),t)}\frac{(\gamma(t)+z(n_2(s,t),t))^2}{\gamma(t)} < 2(1 + s/t - e^{-\gamma(t)})$, and $n^o = 2$ for the remaining values.

Finally, if $s \geq \bar{s}$, we have $1 + s/t \geq h(\gamma(t))$ and $\gamma(t) < 1$ for all $t > t^*(s)$, so that Lemma 4 implies $n^o = 2$ for all values of t. ■

Proof of Proposition 2. This follows immediately from the basic theory of monotone comparative statics, because minimizing ψ is equivalent to maximizing $-\psi$ and $-\psi(n,s,t)$ is strictly submodular (has strictly decreasing differences) in (n,s). For example, one can simply invoke Theorem 2.8.4 in Topkis (1998). ■

Proof of Proposition 3. For $s < s'$, we have $\varphi(s,t) = \psi(n^o(s,t),s,t) \leq \psi(n^o(s',t),s,t) < \psi(n^o(s',t),s',t) = \varphi(s',t)$. Thus, φ is strictly increasing in s.

To argue that $\varphi(s,t)$ is unimodal in t, we show first that $\psi(2,s,t)$ is unimodal in t. Letting $\tau = t/(c_h - c_l)$, we have $\psi(2,s,t) = 2\tau(c_h - c_l)\left(1 - \left(\frac{\tau(1-q)}{q(1-q-\tau)}\right)\right) + 2s$. Note that $t < q(1-q)(c_h - c_l)$ implies $\tau < q(1-q)$ and $1 - q - \tau > (1-q)^2 > 0$. We first compute $\frac{d}{dt}\psi(2,s,t) = 2\frac{\partial}{\partial\tau}\left[\left(1 - \frac{\tau(1-q)}{q(1-q-\tau)}\right)\tau\right] = 2\frac{\tau^2 - 2\tau(1-q) + q(1-q)^2}{q(1-q-\tau)^2}$ and $\frac{d^2}{dt^2}\psi(2,s,t) = -\frac{4}{(c_h-c_l)q}\frac{(1-q)^3+\tau(1-q)}{(1-q-\tau)^3} < 0$. Thus, $\psi(2,s,\cdot)$ is strictly concave, and attains its maximum at $\tau = 1 - q - (1-q)^{3/2}$. Given Proposition 1 (iv) this shows in particular that $\varphi(s,t)$ is unimodal in t if $s \geq \bar{s}$.

Next, by Proposition 1 (i) and Lemma 4 (iii), for $s = 0$ we have $\varphi(s,t) = \min\{\psi(2,0,t), \lim_{n\to\infty}\psi(n,0,t)\}$ and $\lim_{n\to\infty}\psi(n,0,t) = \gamma(t)t$. Since $\frac{d^2}{dt^2}(\gamma(t)t) = \frac{1}{c_h-c_l}\frac{\partial^2}{\partial\tau^2}\left(-\tau\ln\frac{\tau(1-q)}{q(1-q-\tau)}\right) = -\frac{1}{t}\frac{(1-q)^2}{(1-q-\tau)^2} < 0$, we see that $\lim_{n\to\infty}\psi(n,0,t)$ is strictly concave (and attains a maximum on $(0, q(1-q)(c_h - c_l)$, as $\lim_{t\to 0}\gamma(t)t = 0$ and $\lim_{t\to q(1-q)(c_h-c_l)}\gamma(t)t = 0$). Thus, $\varphi(s,t)$ is strictly concave and has a maximum in $(0, q(1-q)(c_h - c_l))$. In particular, it is unimodal in t.

Finally, for $s \in (0,\bar{s})$, we know from Lemma 4 $(ii.b)$ and Proposition 1 (ii) and (iii) that $\varphi(s,t)$ $= \min\{\psi(2,s,t), \psi(n_2(s,t),s,t)\}$ for all values of t, and $\varphi(s,t) = \psi(2,s,t)$ for t close to 0 or close to $\bar{t} = q\,(1-q)\,(c_h - c_l)$. Given that $\psi(2,s,\cdot)$ is strictly concave and attains a maximum on $(0,\bar{t})$, $\varphi(s,\cdot)$ is unimodal if $\psi(n_2(s,t),s,t)$ is either strictly increasing, strictly decreasing or unimodal over the range of values of t for which $\varphi(s,t) = \psi(n_2(s,t),s,t)$ (see Proposition 1 for the characterization this set, which is a subset of $(t^*(s),\bar{t})$, the range of values of t for which $n_2(s,t)$ exists).

By the envelope theorem, we have

$$\frac{d}{dt}\psi(n_2(s,t),s,t) = n_2(s,t)\left(1 - e^{-\frac{\gamma(t,q)}{n_2(s,t)-1}}\left(1 - \frac{t\gamma_t(t,q)}{n_2(s,t)-1}\right)\right) \tag{5.19}$$

where we have added q explicitly as an argument of the function γ, and $\gamma_t = \partial\gamma(t,q)/\partial t$. We need to check that $\frac{d}{dt}\psi(n_2(s,t),s,t)$ is either always positive, always negative, or changes sign exactly once, from positive to negative, as t increases. The presence of the exponent in (5.19) combined with the lack of an explicit expression for n_2 makes an analytical proof infeasible. Therefore, we verify the result numerically. Recall from the proof of Lemma 4 that when $n_2(s,t)$ exists, i.e., for $t \in (t^*(s), \bar{t})$, it is obtained from $z_2(s,t,q)$, the smaller one of the two solutions of the equation $\phi(z, \gamma(t,q), s/t) = 1$, via $z_2(s,t,q) = \frac{\gamma(t,q)}{n_2(s,t)-1}$. Using this, as well as $t\gamma_t(t,q) = -\frac{1-q}{1-q-\tau}$, $\frac{d}{dt}\psi(n_2(s,t),s,t)$ can be rewritten as $n_2(s,t)\left(1 - e^{-z_2(s,t,q)}\left(1 + \frac{1-q}{1-q-\tau}\frac{z_2(s,t,q)}{\gamma(t,q)}\right)\right)$. Using a fine grid of values of q and $\sigma = s/(c_h - c_l)$, we first compute $\tau^*(s,q)$ (note that, in view of the definitions of γ and z_2, it is clear that $\frac{d}{dt}\psi(n_2(s,t),s,t)$ does not depend on $c_h - c_l$, i.e., is a function of σ and $\tau \in (0, q(1-q))$). We then choose a fine grid of $(\tau^*(s,q), q(1-q))$ to compute the corresponding values of $z_2(s,t,q)$ numerically and evaluate $\frac{d}{dt}\psi(n_2(s,t),s,t)$. We find that $\frac{d}{dt}\psi(n_2(s,t),s,t)$ indeed changes sign exactly once, from positive to negative, if q is not too large. If q is large, $\frac{d}{dt}\psi(n_2(s,t),s,t)$ is always negative.[30] To visualize the result, we plot $\frac{d}{dt}\psi(n_2(s,t),s,t)$ (with $c_h - c_l = 1$, i.e., $t = \tau$ and $s = \sigma$ for simplicity) for three values of q in Figure A.1. Hence, $\varphi(s,t)$ is indeed unimodal in t. ■

[30]The Python code and the data matrix are available upon request.

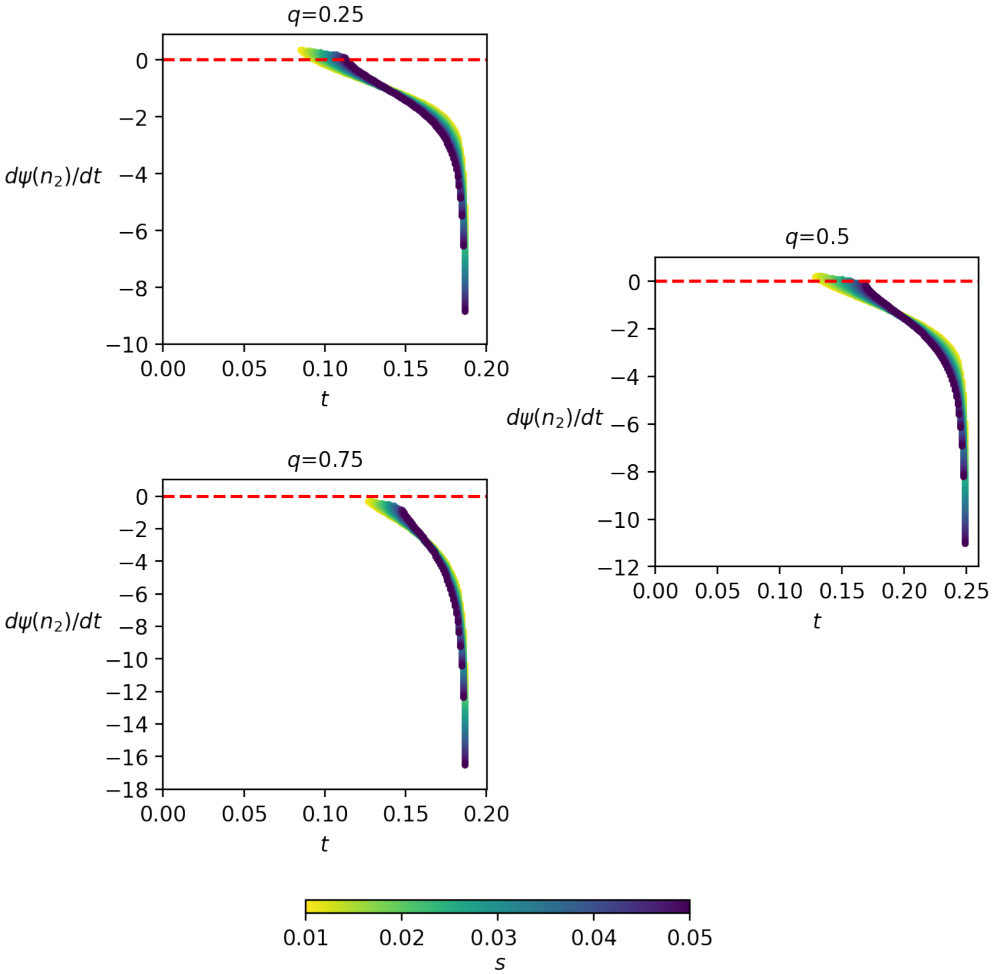

Figure A.1: $\frac{d}{dt}\psi(n_2(s,t),s,t)$ (with $c_h - c_l = 1$, i.e., $t = \tau$ and $s = \sigma$ for simplicity) for three values of q: $q = 0.25$, $q = 0.50$, $q = 0.75$.